NORTH-HOLLAND

Monotonicity Properties of Certain Classes of Norms

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ABSTRACT

Let p be a norm on $\mathbf{K}^n$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. If $S \in \mathbf{K}^{n,n}$ is a nonsingular matrix, let p_S be the norm on $\mathbf{K}^n$, defined by $p_S(x) = p(Sx)$ for all $x \in \mathbf{K}^n$. This note gives some conditions on S for which p_S has a certain monotonicity property, and characterizes p by monotonicity properties of all norms p_S with S in a given group of unitary (orthogonal) matrices. In particular, we obtain the following characterizations: (1) If p is an l_q-norm, $1 < q < \infty$, all matrices S are described for which p_S is weakly monotonic. (2) If for each unitary (orthogonal) matrix $S \in \mathbf{K}^{n,n}$ the norm p_S is quasimonotonic, then p is a positive multiple of the l_2-norm. © Elsevier Science Inc., 1997

INTRODUCTION

In this note $\mathbf{K}$ represents either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers, $\mathbf{K}^n$ is the n-dimensional $\mathbf{K}$-vector space of column vectors $x = (x_1, \dots, x_n)^T$, and $\mathbf{K}^{n,n}$ is the space of all $n \times n$ matrices with entries in $\mathbf{K}$. The standard basis of $\mathbf{K}^n$ is denoted by $\{e_1, \dots, e_n\}$. The space $\mathbf{K}^n$ is endowed with the standard inner product $(x, y) \mapsto y^*x$ (y^* is the conjugate transpose of y) and with the l_2-norm $x \mapsto l_2(x) = (x^*x)^{1/2}$. The symbols $|\cdot|$, $\mathrm{Re}(\cdot)$, $\mathrm{Im}(\cdot)$, $\mathrm{sgn}(\cdot)$, and $\leqslant$ are to be interpreted component-

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wise when applied to vectors or matrices. The Hadamard (or componentwise) product of vectors $x, y \in \mathbf{K}^n$ is denoted by $x \circ y$. We denote the identity matrix of size k by I_k.

A *norm* on $\mathbf{K}^n$ is a functional $p : \mathbf{K}^n \to \mathbf{R}$ satisfying the properties (1) $p(x) \geqslant 0$ for all $x \in \mathbf{K}^n$, with equality iff $x = 0$; (2) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbf{K}$ and $x \in \mathbf{K}^n$; (3) $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in \mathbf{K}^n$. A norm p on $\mathbf{K}^n$ is called *monotonic* if $|x| \leqslant |y|$ implies $p(x) \leqslant p(y)$; *absolute* if $p(x) = p(|x|)$ for all $x \in \mathbf{K}^n$; *subabsolute* if $p(x) \leqslant p(|x|)$ for all $x \in \mathbf{K}^n$; *quasimonotonic* if $0 \leqslant x \leqslant y$ implies $p(x) \leqslant p(y)$; and *weakly monotonic* if $p(x) \leqslant p(y)$ for all $x, y \in \mathbf{K}^n$ such that $x_j = y_j$ for all $j = 1, \dots, n$ except for $j = k$, for which $x_k = 0$.

Monotonicity and absoluteness are equivalent (see [1] or [5]), and imply weak monotonicity and subabsoluteness. It is shown in [6] that the properties of weak monotonicity and subabsoluteness are independent and both imply quasimonotonicity. A simple proof that a subabsolute norm is quasimonotonic can be derived directly from the proof of Theorem 5.5.10 of [5]. Weak monotonicity is known also as *orthant monotonicity (see [10]), and coincides for $\mathbf{K} = \mathbf{R}$ with orthant monotonicity introduced in [2].

Let p be a norm on $\mathbf{K}^n$, and let $S \in \mathbf{K}^{n,n}$ be a nonsingular matrix. Then the functional $p_S : x \mapsto p(Sx)$ is a norm on $\mathbf{K}^n$ that is called the *S-transform* of p. In this paper we study the conditions on S for which the norm p_S has one of the above-mentioned monotonicity properties. In particular, when p is an l_q-norm, $1 < q < \infty$, we obtain a description of all matrices S for which p_S is weakly monotonic. As a consequence we get a special case of characterizations given in [3] and [4]. We consider also the following problem: Characterize p so that p_S has a certain monotonicity property for all S in a given subgroup of the group of all unitary (orthogonal) matrices on $\mathbf{K}^n$. In particular, we show that all transforms p_S with unitary (orthogonal) matrices S are quasimonotonic if and only if p is a positive multiple of the l_2-norm. Similar problems are considered also in a survey [7].

Our proofs are unified in the sense that the real and the complex cases are covered simultaneously. They are mainly based on geometry and techniques of mathematical analysis, so we recall some additional definitions and notation.

Let p be a norm on $\mathbf{K}^n$. The *subdifferential* of p at $x \in \mathbf{K}^n$ is the set

$$\partial p(x) := \{v \in \mathbf{K}^n : p(x + y) - p(x) \geqslant \operatorname{Re}(v^* y) \qquad \text{for all} \quad y \in \mathbf{K}^n\}.$$

If we adopt the standard identification of $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ (see for example [2]), this definition of the subdifferential and its following properties can be found in [9].

The subdifferential $\partial p(x)$ is a nonempty, convex, and compact subset of $\mathbf{K}^n$. It is closely related to the differential of p at x. Namely, $\partial p(x)$ is a one-point set $\{v\}$ if and only if the $\mathbf{R}$-differential of p at x equals v, i.e.,

$$\lim_{y \to 0} \frac{p(x+y) - p(x) - \operatorname{Re}(v^* y)}{p(y)} = 0.$$

In this case we shall write $\partial p(x) = v$. If p is an l_q-norm, $1 < q < \infty$, then p is $\mathbf{R}$-differentiable at each nonzero $x \in \mathbf{K}^n$; moreover,

$$\partial p(x) = p(x)^{1-q} \operatorname{sgn}(x) \circ |x|^{q-1},$$

where $(|x|^{q-1})_j = |x_j|^{q-1}$ for $j = 1, \ldots, n$.

For each $x, y \in \mathbf{K}^n$ there exists a directional derivative

$$p'(x; y) = \lim_{\lambda \downarrow 0} \frac{p(x + \lambda y) - p(x)}{\lambda},$$

which is related to the subdifferential of p by

$$p'(x; y) = \sup\{\operatorname{Re}(u^* y) : u \in \partial p(x)\}.$$

RESULTS

LEMMA 1. *Let p be a norm on $\mathbf{K}^n$, and let $x, y \in \mathbf{K}^n$ be given. Then*

$$p(x) \leqslant p(x + \lambda y) \qquad \textit{for all} \quad \lambda \in \mathbf{K} \tag{1}$$

if and only if there exists $u \in \partial p(x)$ such that $u^* y = 0$;

$$p(x) \leqslant p(x + \lambda y) \qquad \textit{for all} \quad \lambda \in \mathbf{R}^+ \tag{2}$$

if and only if there exists $u \in \partial p(x)$ such that $\operatorname{Re}(u^* y) \geqslant 0$; *and*

$$p(x + \lambda y) \leqslant p(x + |\lambda| y) \qquad \textit{for all} \quad \lambda \in \mathbf{K} \tag{3}$$

implies that there exists $u \in \partial p(x)$ such that $u^* y \geqslant 0$.

Proof. The first equivalence follows easily from [10, Proposition 2.3].

To prove the second equivalence, note first that by [9, Theorem 23.1] (2) is equivalent to $p'(x;\, y) \geqslant 0$, and therefore to

$$\sup\{\operatorname{Re}(u^* y) : u \in \partial p(x)\} \geqslant 0.$$

Since $\partial p(x)$ is compact, (2) holds if and only if $\operatorname{Re}(u^* y) \geqslant 0$ for some $u \in \partial p(x)$.

Obviously (3) is equivalent to

$$\frac{p(x + |\lambda|(\operatorname{sgn} \lambda) y) - p(x)}{|\lambda|} \leqslant \frac{p(x + |\lambda| y) - p(x)}{|\lambda|} \qquad \text{for all} \quad \lambda \in \mathbf{K} \setminus \{0\};$$

hence it implies that

$$p'(x; \mu y) \leqslant p'(x; y) \qquad \text{for all} \quad \mu \in \mathbf{K}, \quad |\mu| = 1.$$

It follows that

$$\sup\{\operatorname{Re}(\mu u^* y) : u \in \partial p(x)\} \leqslant \sup\{\operatorname{Re}(u^* y) : u \in \partial p(x)\}$$

for all $\mu \in \mathbf{K}$ satisfying $|\mu| = 1$. Using the fact that

$$\sup\{\operatorname{Re}(\mu\nu) : \mu \in \mathbf{K}, |\mu| = 1\} = |\nu|, \qquad \nu \in \mathbf{K},$$

we get

$$\sup\{|u^* y| : u \in \partial p(x)\} \leqslant \sup\{\operatorname{Re}(u^* y) : u \in \partial p(x)\}.$$

Since $\partial p(x)$ is compact, there exists a $u \in \partial p(x)$ such that $u^* y \geqslant 0$. ■

THEOREM 2. *Let p be a norm on $\mathbf{K}^n$, and let $S \in \mathbf{K}^{n,n}$ be a given nonsingular matrix. Then*

(1) *p_S is weakly monotonic if and only if for each $x, y \in \mathbf{K}^n$ satisfying $x \circ y = 0$ there exists a $u \in \partial p(Sx)$ such that $u^* Sy = 0$;*

(2) *p_S is quasimonotonic if and only if for each $x, y \in \mathbf{K}^n$ satisfying $x \geqslant 0$, $y \geqslant 0$, and $x \circ y = 0$ there exists a $u \in \partial p(Sx)$ such that* $\operatorname{Re}(u^* Sy) \geqslant 0$;

(3) *If p_S is subabsolute, then for each $x, y \in \mathbf{K}^n$ satisfying $x \geqslant 0$, $y \geqslant 0$, and $x \circ y = 0$ there exists a $u \in \partial p(Sx)$ such that $u^* Sy \geqslant 0$.*

Proof. Observe that

$$\partial p_S(z) = S^* \,\partial p(Sz) \qquad \text{for all} \quad z \in \mathbf{K}$$

[9, Theorem 23.9], and combine Lemma 1 with the following simple facts:

(1) p_S is weakly monotonic if and only if $p_S(x) \leqslant p_S(x+y)$ for all $x, y \in \mathbf{K}^n$ satisfying $x \circ y = 0$;

(2) p_S is quasimonotonic if and only if $p_S(x) \leqslant p_S(x+y)$ for all $x, y \in \mathbf{K}^n$ satisfying $x \geqslant 0$, $y \geqslant 0$, and $x \circ y = 0$;

(3) If p_S is subabsolute, then $p_S(x + \lambda y) \leqslant p_S(|x + \lambda y|) = p_S(x + |\lambda| y)$ for all $x, y \in \mathbf{K}^n$ satisfying $x \geqslant 0$, $y \geqslant 0$, and $x \circ y = 0$. ∎

COROLLARY 3. *Let $S \in \mathbf{K}^{n,n}$ be a nonsingular matrix, let p be an $\mathbf{R}$-differentiable norm on $\mathbf{K}^n$, and define $\partial p(S) := [\partial p(Se_1), \dots, \partial p(Se_n)] \in \mathbf{K}^{n,n}$. Then:*

(1) *If p_S is weakly monotonic, then $S^* \,\partial p(S)$ is diagonal.*

(2) *If p_S is quasimonotonic, then* $\mathrm{Re}[S^* \,\partial p(S)] \geqslant 0$.

(3) *If p_S is subabsolute, then $S^* \,\partial p(S) \geqslant 0$.*

Proof. For every pair of different indices i, j take $x = e_i$, $y = e_j$ in Theorem 2, and note that $u^* \,\partial p(u) \geqslant 0$ for all $u \in \mathbf{K}^n$. ∎

REMARK 4. It follows easily from Theorem 2 that if $n = 2$ then the necessity condition of (1) [respectively (2)] in Corollary 3 is also sufficient for p_S to be weakly monotonic [respectively quasimonotonic].

Since there exists a norm p on $\mathbf{R}^3$ that is not quasimonotonic even though $\partial p(I)$ is diagonal and $\partial p(I) \geqslant 0$, the converse of (1), (2), and (3) does not hold in general. Nevertheless, for the l_2-norm we have the following result.

THEOREM 5. *Let $S \in \mathbf{K}^{n,n}$ be a nonsingular matrix, and let p be the l_2-norm on $\mathbf{K}^n$. Then*

(1) *p_S is monotonic, absolute, or weakly monotonic if and only if $S^* S$ is diagonal;*

(2) *p_S is quasimonotonic if and only if* $\mathrm{Re}(S^* S) \geqslant 0$;

(3) *p_S is subabsolute if and only if $S^* S \geqslant 0$.*

Proof. Note that $\partial p(x) = x$ for all $x \in \mathbf{K}^n$ and therefore $\partial p(S) = S$.

(1): If S^*S is diagonal, then $|x|^* S^* S|x| = x^* S^* Sx$ for each $x \in \mathbf{K}^n$, and hence p_S is absolute. Since monotonicity is equivalent to absoluteness and implies weak monotonicity, Corollary 3 completes the proof of (1).

(2): If $\operatorname{Re}(S^*S) \geqslant 0$, then $\operatorname{Re}(y^*S^*Sx) \geqslant 0$, and therefore $p_S(x) \leqslant p_S(x+y)$ for all vectors $x, y \in \mathbf{K}^n$ satisfying $x \geqslant 0$, $y \geqslant 0$; p_S is therefore quasimonotonic. The converse follows from Corollary 3.

(3): If $S^*S \geqslant 0$, then $|x|^* S^* S|x| \geqslant x^* S^* Sx$ for each $x \in \mathbf{K}^n$, so p_S is subabsolute. The converse follows from Corollary 3. ∎

If p is an l_q-norm with $1 < q \neq 2$, a matrix S for which the p_S-transform is weakly monotonic or monotonic can be characterized as follows.

THEOREM 6. *Let $S \in \mathbf{K}^{n,n}$ be nonsingular, let p be the l_q-norm on $\mathbf{K}^n$, and suppose $1 < q \in \mathbf{R}$, $q \neq 2$. Then:*

(1) *p_S is weakly monotonic if and only if there exist permutation matrices $P, Q \in \mathbf{K}^{n,n}$ and a nonsingular diagonal matrix $D \in \mathbf{K}^{n,n}$ such that $PSQ = S_0 D$ with*

$$S_0 = C(\nu_1) \oplus \cdots \oplus C(\nu_r) \oplus I_{n-2r}, \tag{4}$$

where for $j = 1, \dots, r$

$$C(\nu_j) \equiv \begin{bmatrix} 1 & 1 \\ \nu_j & -\nu_j \end{bmatrix}, \qquad \nu_j \in \mathbf{K}, \quad |\nu_j| = 1, \quad 0 \leqslant \arg \nu_j < \pi.$$

(2) *If $\mathbf{K} = \mathbf{R}$, then p_S is monotonic or absolute if and only if it is weakly-monotonic. In this case, $S_0 = C(1) \oplus \cdots \oplus C(1) \oplus I_{n-2r}$. If $\mathbf{K} = \mathbf{C}$, then p_S is monotonic or absolute if and only if there exists a permutation matrix $R \in \mathbf{K}^{n,n}$ such that RS is diagonal.*

Proof. (1): Let p_S be weakly monotonic. It follows from Theorem 2 that for each $j \in \{1, \dots, n\}$

$$(Sx)^* \, \partial p(Se_j) = 0 \qquad \text{for all} \quad x \in e_j^{\perp}.$$

Since S is nonsingular, this implies that

$$S\left(e_j^{\perp}\right) = \left[\partial p\left(Se_j\right)\right]^{\perp}, \qquad j = 1, \dots, n. \tag{5}$$

Using Theorem 2 again, we get $(Se_j)^* \, \partial p(Sx) = 0$ for all $x \in e_j^{\perp}$, and therefore (5) ensures that

$$\left(Se_j\right)^* \, \partial p(v) = 0 \qquad \text{for all} \quad v \in \left[\partial p\left(Se_j\right)\right]^{\perp}.$$

Since $\partial p(w) = p(w)^{1-q} \operatorname{sgn}(w) \circ |w|^{q-1}$, $0 \neq w \in \mathbf{K}^n$, it follows that for each $u = Se_j$, $j = 1, \dots, n$, we have

$$v^*\left(\operatorname{sgn}(u) \circ |u|^{q-1}\right) = 0 \quad \Rightarrow \quad u^*\left(\operatorname{sgn}(v) \circ |v|^{q-1}\right) = 0. \tag{6}$$

We claim that

(a) $u_k u_l \neq 0$ implies $|u_k| = |u_l|$;
(b) $u_k u_l u_m = 0$ for all choices of different indices k, l, and m.

To show (a), suppose that $u_k u_l \neq 0$, $k \neq l$, and take

$$v = \operatorname{sgn}(\bar{u}_l)\,|u_l|^{q-1} e_k - \operatorname{sgn}(\bar{u}_k)\,|u_k|^{q-1} e_l.$$

Then $v^*(\operatorname{sgn}(u) \circ |u|^{q-1}) = 0$, and therefore by (6)

$$\begin{aligned} 0 &= \bar{u}_k \operatorname{sgn}(v_k)\,|v_k|^{q-1} + \bar{u}_l \operatorname{sgn}(v_l)\,|v_l|^{q-1} \\ &= \bar{u}_k \operatorname{sgn}(\bar{u}_l)\,|u_l|^{(q-1)^2} - \bar{u}_l \operatorname{sgn}(\bar{u}_k)\,|u_k|^{(q-1)^2}. \end{aligned}$$

It follows that $|u_k|^{q(q-2)} = |u_l|^{q(q-2)}$; hence $|u_k| = |u_l|$ $(q \neq 2)$.

To prove (b), suppose that $u_k u_l u_m \neq 0$ for some given choice of different indices k, l, and m, and take

$$v = \operatorname{sgn}(u_k)\, e_k + \operatorname{sgn}(u_l)\, e_l - 2\operatorname{sgn}(u_m)\, e_m.$$

By (a), $v^*\,(\operatorname{sgn}(u) \circ |u|^{q-1}) = 0$, and hence (6) implies that $u^*(\operatorname{sgn}(v) \circ |v|^{q-1}) = 0$. An easy calculation shows that $|u_k| + |u_l| - 2^{q-1}|u_m| = 0$ and consequently $2^{q-1} = 2$. This is a contradiction, since $q \neq 2$.

It follows from (a) and (b) that for each $j \in \{1, \dots, n\}$ there exist different indices $\alpha(j)$, $\beta(j)$ such that

$$Se_j = \lambda_j(e_{\alpha(j)} + \mu_j e_{\beta(j)})$$

for some λ_j, $\mu_j \in \mathbf{K}$ satisfying $\lambda_j \neq 0$ and $|\mu_j| = 1$ or $\mu_j = 0$. This implies that

$$\operatorname{sgn}(Se_j) \circ |Se_j|^{q-1} = |\lambda_j|^{q-2} Se_j, \qquad j = 1, \dots, n;$$

Corollary 3 ensures that $\{Se_1, \dots, Se_n\}$ is an orthogonal basis of $\mathbf{K}^n$.

Let J_1 be the set of all indices j satisfying $|\mu_j| = 1$, and let J_2 be the set of all indices j satisfying $\mu_j = 0$. If k and l are different indices of J_1, then exactly one of the following two possibilities occurs:

$$\{\alpha(k), \beta(k)\} \cap \{\alpha(l), \beta(l)\} = \varnothing, \tag{7}$$

or

$$\alpha(k) = \alpha(l), \qquad \beta(k) = \beta(l), \quad \text{and} \quad \mu_k + \mu_l = 0. \tag{8}$$

If k and l are different indices in J_2, then $\alpha(k) \neq \alpha(l)$. If $k \in J_1$ and $l \in J_2$, then $\alpha(l) \neq \alpha(k)$ and $\alpha(l) \neq \beta(k)$. It follows that for each $k \in J_1$ there exists a unique $l \in J_1$ satisfying (8) and such that all other indices $l \in J_1 \setminus \{k\}$ satisfy (7). It is now routine to verify that there exist permutation matrices P, Q, and a nonsingular diagonal matrix D such that $PSQ = S_0 D$, where S_0 satisfies (4) (and $2r = |J_1|$).

To prove the converse, suppose that S is of the prescribed form (4). Since p is permutation invariant, and $x \circ y = 0$ is equivalent to $(DQ^{-1}x) \circ (DQ^{-1}y) = 0$, p_S is weakly monotonic if and only if p_{S_0} is weakly monotonic. Using (4), it suffices to prove that for each $\nu \in \mathbf{K}$ satisfying $|\nu| = 1$ the $C(\nu)$-transform $\tilde{p}_{C(\nu)}$ of the l_q-norm $\tilde{p}$ on $\mathbf{K}^2$ is weakly monotonic. An easy computation shows that $C(\nu)^* \, \partial\tilde{p}(C(\nu)) = C(\nu)^* C(\nu) = I_2$, so Remark 4 ensures that the norm $\tilde{p}_{C(\nu)}$ is weakly monotonic.

(2): Since $S_0 = C(\nu_1) \oplus \cdots \oplus C(\nu_r) \oplus I_{n-2r}$, the proof of (2) reduces to the two-dimensional case with $C(\nu_j)$ instead of S_0.

If $\mathbf{K} = \mathbf{R}$, then by part (1) $S_0 = C(1) \oplus \cdots \oplus C(1) \oplus I_{n-2r}$. The case with $C(1)$ is then easy. If $\mathbf{K} = \mathbf{C}$, it suffices to show that $r = 0$. To this end, assume that $r \geqslant 1$, put

$$E = C(\nu)\,\mathrm{diag}(a, b) \in \mathbf{C}^{2,2}, \qquad |\nu| = 1, \quad ab \neq 0,$$

and suppose that the E-transform of the l_q-norm on $\mathbf{C}^2$ is absolute. Applying this absolute norm to $[1, e^{i\phi}]^T$, we get

$$|a + be^{i\phi}|^q + |a - be^{i\phi}|^q = |a + b|^q + |a - b|^q \qquad \text{for all} \quad \phi \in \mathbf{R}.$$

Because $q \neq 2$, it is easy to see that this is impossible, and hence $r = 0$. ∎

The characterization (2) in Theorem 6 is a special case of [4, Theorem 3.2(a)], where a quite different approach is used. A similar result is obtained in [3] for rectangular matrices. It would be interesting to obtain analogous results for norms that are quasimonotonic or subabsolute.

One of our main results characterizes the l_2-norm as follows.

THEOREM 7. *Let p be a given norm on $\mathbf{K}^n$. The following three conditions are equivalent*:

(a) *p_S is monotonic for each nonsingular matrix $S \in \mathbf{K}^{n,n}$ such that S^*S is diagonal.*

(b) *p_S is quasimonotonic for each unitary (orthogonal) matrix $S \in \mathbf{K}^{n,n}$.*

(c) *There exists a real $a > 0$ such that $p(x) = al_2(x)$ for all $x \in \mathbf{K}^n$.*

Proof. We give the proof for $\mathbf{K} = \mathbf{C}$; the same argument works for $\mathbf{K} = \mathbf{R}$ if one replaces all references to unitary matrices by references to real orthogonal matrices.

The implication (a) ⇒ (b) is clear.

(b) ⇒ (c): Suppose that $n > 1$, fix a nonzero $x_0 \in \mathbf{K}^n$ and take any $x \in \mathbf{K}^n$ such that x_0 and x are linearly independent. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of $\mathbf{K}^n$ such that

$$v_1 = \frac{x_0}{l_2(x_0)}, \qquad v_2 = \frac{x - \mu x_0}{l_2(x - \mu x_0)}, \quad \mu \in \mathbf{K},$$

and let $V \in \mathbf{K}^{n,n}$ be defined by $Ve_j = v_j$ for $j = 1, \dots, n$. Then there exists a unitary $U \in \mathbf{K}^{n,n}$ satisfying

$$Ux_0 = \lambda x, \qquad \lambda = \frac{l_2(x_0)}{l_2(x)},$$

and $V^*UV = W \oplus I_{n-2}$, where

$$W = \begin{bmatrix} \tau & -t \\ t & \bar{\tau} \end{bmatrix}, \qquad \tau = \frac{x_0^* x}{l_2(x_0) l_2(x)}, \quad t = \left(1 - |\tau|^2\right)^{1/2} > 0.$$

Note that the eigenvalues of W (in $\mathbf{C}$) are of the form $e^{i\phi}, e^{-i\phi}$, $0 < \phi < \pi$. For each natural number m let $W_m \in \mathbf{K}^{2,2}$ be a unitary matrix with eigenvalues $e^{i\phi/m}, e^{-i\phi/m}$, such that $(W_m)^m = W$. Then

$$U_m = V(W_m \oplus I_{n-2})V^* \in \mathbf{K}^{n,n}$$

satisfies $(U_m)^m = U$.

Take any $z \in \mathbf{K}^2$ satisfying $z^*z = 1$, fix m, note that $|z^*W_m z| \leqslant 1$, and choose an orthonormal basis $\{w_1, w_2\}$ of $\mathbf{C}^2$ such that

$$W_m w_1 = e^{i\phi/m} w_1, \qquad W_m w_2 = e^{-i\phi/m} w_2.$$

Then $z = \xi_1 w_1 + \xi_2 w_2$ for some $\xi_1, \xi_2 \in \mathbf{C}$, and $|\xi_1|^2 + |\xi_2|^2 = 1$. It follows that

$$|z^* U_m z|^2 = \cos^2 \frac{\phi}{m} + \left(|\xi_1|^2 - |\xi_2|^2\right)^2 \sin^2 \frac{\phi}{m}.$$

Thus $|z^* W_m z| \geqslant |\cos(\phi/m)|$, and therefore

$$\lim_{m \to \infty} |z^* W_m z|^m = 1.$$

Since

$$x_0^* U_m x_0 = l_2(x_0)^2 e_1^* (W_m \oplus I_{n-2}) e_1,$$

it follows that the sequence

$$\lambda_m = \frac{x_0^* x_0}{x_0^* U_m x_0}, \qquad m = 2, 3, \dots,$$

satisfies $\lim_{m \to \infty} |\lambda_m|^m = 1$.

For each $m > 2$ put $x_0(m) = x_0$ and

$$x_k(m) = \lambda_m U_m x_{k-1}(m), \qquad k = 1, \dots, m.$$

Then $x_m := x_m(m)$ satisfies

$$x_m = (\lambda_m)^m (U_m)^m x_0 = (\lambda_m)^m U x_0 = (\lambda_m)^m \lambda x,$$

and therefore by continuity of p

$$\lim_{m \to \infty} p(x_m) = \lambda p(x). \tag{9}$$

We claim that $p(x_0) \leqslant p(x_m)$. An easy computation shows that

$$x_{k-1}(m)^* [x_k(m) - x_{k-1}(m)] = |\lambda_m|^2 x_{k-2}(m)^* [x_{k-1}(m) - x_{k-2}(m)],$$

and hence

$$x_{k-1}(m)^* [x_k(m) - x_{k-1}(m)] = |\lambda_m|^{2(k-1)} x_0(m)^* [x_1(m) - x_0(m)] = 0$$

for $k = 1, \dots, m$. Since x_0 and x are linearly independent, the equality $(U_m)^m x_0 = \lambda x$ shows that $x_{k-1}(m) \neq 0$ and $x_k(m) - x_{k-1}(m) \neq 0$.

Let

$$\frac{x_{k-1}(m)}{l_2(x_{k-1}(m))}, \qquad \frac{x_k(m) - x_{k-1}(m)}{l_2(x_k(m) - x_{k-1}(m))}$$

be the first and the second column of a unitary $S_k(m) \in \mathbf{K}^{n,n}$. Then

$$0 \leqslant S_k(m)^{-1} x_{k-1}(m) \leqslant S_k(m)^{-1} x_k(m),$$

and quasi-monotonicity of the norm $p_{S_k(m)}$ implies that

$$\begin{aligned} p(x_{k-1}(m)) &= p_{S_k(m)}\left(S_k(m)^{-1} x_{k-1}^m\right) \\ &\leqslant p_{S_k(m)}\left(S_k(m)^{-1} x_k(m)\right) = p(x_k(m)). \end{aligned}$$

This yields

$$p(x_0) = p(x_0(m)) \leqslant p(x_1(m)) \leqslant \cdots \leqslant p(x_m(m)) = p(x_m),$$

and the claim follows.

Using (9), we get

$$p(x_0) \leqslant \lim_{m\to\infty} p(x_m) = \lambda p(x) = \frac{l_2(x_0)}{l_2(x)} p(x).$$

Interchanging x_0 and x gives the reverse inequality, so

$$p(x) = \frac{p(x_0)}{l_2(x_0)} l_2(x),$$

and (c) follows easily.

The implication (c) $\Rightarrow$ (a) is a part of Theorem 5. ■

The group of all linear isometries of the space $(\mathbf{K}^n, l_2)$ consists of all unitary matrices when $\mathbf{K} = \mathbf{C}$, and all real orthogonal matrices when $\mathbf{K} = \mathbf{R}$. Theorem 7 ensures that this group is rich enough to make possible a monotonicity characterization of l_2-norm. It is tempting to conjecture that a similar characterization holds for an arbitrary l_q-norm on $\mathbf{K}^n$, $1 \leqslant q \leqslant \infty$ [replacing unitary matrices in (b) by linear isometries of $(\mathbf{K}^n, l_q)$ and the l_2-norm in (c) by the l_q-norm]. The group G of all linear isometries of the space $(\mathbf{K}^n, l_q)$, where $1 \leqslant q \leqslant \infty$ and $q \neq 2$, consists of all generalized permutation matrices (see [8] for a short proof), hence it is independent of q. Therefore, we cannot expect that quasimonotonicity of p_S for each $S \in G$ implies that p is a positive multiple of the l_q-norm. However, we have the following result.

THEOREM 8. *Let G be the group of all generalized permutation matrices, and let G_0 be its subgroup consisting of all diagonal unitary (orthogonal) matrices $E \in \mathbf{K}^{n,n}$. For a norm p on $\mathbf{K}^n$, the following three conditions are equivalent:*

(a) *p is weakly monotonic.*
(b) *p_S is weakly monotonic for each $S \in G$ (or each $S \in G_0$).*
(c) *p_S is quasimonotonic for each $S \in G$ (or each $S \in G_0$).*

Proof. (a) $\Rightarrow$ (b): Suppose that p is weakly monotonic, and take any $S \in G$. By [8] there exists a permutation matrix $P \in \mathbf{K}^{n,n}$ and some $E \in G_0$ such that $S = PE$. If $D \in \mathbf{K}^{n,n}$ satisfies $0 \leqslant D \leqslant I_n$, then $ED = DE$ and $0 \leqslant D_1 = PDP^{-1} \leqslant I_n$. It follows by [6] that

$$p_S(Dx) = p(D_1(PEx)) \leqslant p(PEx) = p_S(x)$$

holds for all $x \in \mathbf{K}^n$, and hence p_S is weakly monotonic.

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (a): Suppose that p satisfies (c), take any $x \in \mathbf{K}^n$, and choose $E \in G_0$ such that $E|x| = x$. If $D \in \mathbf{K}^{n,n}$ satisfies $0 \leqslant D \leqslant I_n$, then $0 \leqslant D|x| \leqslant |x|$, and therefore by [6]

$$p(Dx) = p_E(DE^{-1}x) \leqslant p_E(|x|) = p(x).$$

It follows that p is weakly monotonic. ■

A similar characterization holds for monotonic norms.

THEOREM 9. *Let G be the group of all generalized permutation matrices, and let G_0 be its subgroup consisting of all diagonal unitary (orthogonal) matrices $E \in \mathbf{K}^{n,n}$. For a norm p on $\mathbf{K}^n$ the following three conditions are equivalent*:

(a) *p is monotonic.*

(b) *p_S is monotonic for each $S \in G$ (or each $S \in G_0$).*

(c) *p_S is subabsolute for each $S \in G$ (or each $S \in G_0$).*

Proof. We shall use the fact that a norm is monotonic if and only if it is absolute.

(a) $\Rightarrow$ (b): Suppose p is monotonic, take any $S \in G$, and note that $|S|x|| = |Sx|$ for all $x \in \mathbf{K}^n$. Then

$$p_S(|x|) = p(S|x|) = p(|S|x||) = p(|Sx|) = p(Sx) = p_S(x)$$

for all $x \in \mathbf{K}^n$, and hence (b) follows.

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (a): Suppose p satisfies (c), take any $x \in \mathbf{K}^n$, and choose $E \in G_0$ such that $E|x| = x$. Then

$$p(|x|) = p_E(E^{-2}x) \leqslant p_E(|E^{-2}x|) = p_E(|x|) = p(x).$$

Since p is also subabsolute, we get $p(x) = p(|x|)$, and hence (a) follows. ■

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